Chapter 11 Series Part 2

1. (a) The sum of the first two terms of a geometric progression is 10 and the third term is 9.

(i) Find the possible values of the common ratio and the first term. [5]

$a + ar = 10$

$a(1+r) = 10$

$ar^2 = 9$

$a = \frac{9}{r^2}$

$\frac{9}{r^2}(1+r) = 10$

$9 + 9r = 10r^2$

$10r^2 - 9r - 9 = 0$

$(2r-3)(5r+3) = 0$

$r = \frac{3}{2}$ or $r = -\frac{3}{5}$

$a = \frac{9}{r^2}$
$= 4$

$a = 25$

(ii) Find the sum to infinity of the convergent progression. [1]

$-1 < r < 1$

$r = -\frac{3}{5}$

$S_\infty = \frac{a}{1-r} = \frac{25}{1+\frac{3}{5}} = \frac{125}{8} = 15\frac{5}{8}$

(b) In an arithmetic progression, $u_1 = -10$ and $u_4 = 14$. Find $u_{100} + u_{101} + u_{102} + \ldots + u_{200}$, the sum of the 100th to the 200th terms of the progression.

[4]

$a = -10$

$a + 3d = 14$

$-10 + 3d = 14$

$3d = 24$

$d = 8$

$a = u_{100} = a + 99d$

$= 782$

$S_{101} = \frac{101}{2}(2a + 100d)$

$= \frac{101}{2}(1564 + 800)$

$= 119382$

2. (a) An arithmetic progression has a second term of -14 and a sum to 21 terms of 84. Find the first term and the 21st term of this progression.

[5]

$a+d=-14$

$S_{21}=\frac{21}{2}(2a+20d)=84$

$$\begin{array}{r} 21a+210d=84 \\ -\quad 21a+21d=-294 \\ \hline 189d=378 \end{array}$$

$d=2$

$a+2=-14$

$a=-16$

21^{st} term $= a+20d$

$=-16+40$

$=24$

(b) A geometric progression has a second term of $27p^2$ and a fifth term of p^5. The common ratio, r, is such that $0 < r < 1$.

(i) Find r in terms of p. [2]

$ar = 27p^2$ $\quad$ $ar^4 = p^5$ $\quad$ $r^3 = \frac{p^5}{27p^2}$ $\quad$ $r = \frac{p}{3}$

$a = \frac{27p^2}{r}$ $\quad$ $\frac{27p^2}{r} \times r^4 = p^5$ $\quad$ $r^3 = \frac{p^3}{27}$

(ii) Hence find, in terms of p, the sum to infinity of the progression. [3]

$a = \frac{27p^2}{p/3} = 27p^{\cancel{2}} \times \frac{3}{\cancel{p}}$

$= 81p$

$S_\infty = \frac{a}{1-r} = \frac{81p}{1-\frac{p}{3}} = \frac{81p}{\frac{3-p}{3}} = 81p \times \frac{3}{3-p}$

$= \frac{243p}{3-p}$

(iii) Given that the sum to infinity is 81, find the value of p. [2]

$\frac{243p}{3-p} = 81$

$243p = 243 - 81p$

$324p = 243$

$p = \frac{3}{4}$

3.(a) An arithmetic progression has a first term of 7 and a common difference of 0.4. Find the least number of terms so that the sum of the progression is greater than 300.

[4]

$a = 7$

$d = 0.4$

$S_n > 300$

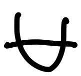
$\frac{n}{2}(14 + (n-1)0.4) > 300$

$n(14 + 0.4n - 0.4) > 600$

$n(13.6 + 0.4n) > 600$

$13.6n + 0.4n^2 - 600 > 0$

$4n^2 + 136n - 6000 > 0$



$n > 25.3 \quad \text{or} \quad n < -59.3$

$\therefore n = 26$

(b) The sum of the first two terms of a geometric progression is 9 and its sum to infinity is 36. Given that the terms of the progression are positive, find the common ratio.

[4]

$a + ar = 9 \rightarrow a(1+r) = 9$

$a = \frac{9}{1+r}$

$\frac{a}{1-r} = 36$

$a = 36(1-r)$

$\frac{9}{1+r} = 36(1-r)$

$9 = 36(1-r)(1+r)$

$9 = 36(1-r^2)$

$1 - r^2 = \frac{1}{4}$

$r^2 = 1 - \frac{1}{4}$

$r^2 = \frac{3}{4}$

$r = \frac{\sqrt{3}}{2}$

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4.(a) The first 5 terms of a sequence are given below.

4 -2 1 -0.5 0.25

(i) Find the 20th term of the sequence.

[2]

$a = 4$

$r = \frac{-2}{4} = -\frac{1}{2}$

20^{th} term $= ar^{19}$

$= 4 \times \left(-\frac{1}{2}\right)^{19}$

$= -\frac{1}{131072}$

(ii) Explain why the sum to infinity exists for this sequence and find the value of this sum.

[2]

$r = -\frac{1}{2}$

sum to infinity exists when $-1 < r < 1$ and since the r for this sequence is $-\frac{1}{2}$, sum to infinity exists.

$S_\infty = \frac{a}{1-r} = \frac{4}{1+\frac{1}{2}} = \frac{4}{\frac{3}{2}} = \frac{8}{3}$

(b) The tenth term of an arithmetic progression is 15 times the second term. The sum of the first 6 terms of the progression is 87.

(i) Find the common difference of the progression.

[4]

$a+9d = 15(a+d)$

$a+9d = 15a+15d$

$0 = 14a+6d$

$7a+3d=0$ — ①

$35a+15d=0$

$-\ 6a+15d=87$

$29a = -87$

$a=-3$

$-21+3d=0$

$3d=21$

$d=7$

$S_6 = 87$

$\frac{6}{2}(2a+5d)=87$

$2a+5d=29$ — ②

(ii)For this progression, the *n*th term is 6990. Find the value of *n*.

[3]

$a+(n-1)d = 6990$

$-3+7n-7 = 6990$

$7n-10 = 6990$

$7n = 7000$

$n = 1000$

5.(a) An arithmetic progression has a second term of 8 and a fourth term of 18. Find the least number of terms for which the sum of this progression is greater than 1560.

[6]

$$a + d = 8$$
$$a + 3d = 18$$
$$-2d = -10$$
$$d = 5$$

$$a + d = 8$$
$$a = 3$$

$$S_n > 1560$$

$$\frac{n}{2}(2a + (n-1)d) > 1560$$

$$\frac{n}{2}(6 + (n-1)5) > 1560$$

$$n(6 + 5n - 5) > 3120$$

$$n(1 + 5n) > 3120$$

$$n + 5n^2 - 3120 > 0$$

$$5n^2 + n - 3120 > 0$$

$$n > 24.9 \quad \text{or} \quad n < -25.1$$

$$\therefore n = 25$$

(b) A geometric progression has a sum to infinity of 72. The sum of the first 3 terms of this progression is $\frac{333}{8}$.

(i) Find the value of the common ratio. [5]

$$S_\infty = \frac{a}{1-r}$$

$$\frac{a}{1-r} = 72$$

$$a = 72(1-r)$$

$$S_3 = \frac{a(1-r^3)}{1-r}$$

$$\frac{333}{8} = \frac{72 \times \cancel{(1-r)}(1-r^3)}{\cancel{1-r}}$$

$$\frac{37}{64} = 1-r^3$$

$$r^3 = 1 - \frac{37}{64}$$

$$r^3 = \frac{27}{64}$$

$$r = \frac{3}{4}$$

(ii) Hence find the value of the first term. [1]

$$a = 72(1-r)$$

$$= 72\left(1-\frac{3}{4}\right)$$

$$= 72 \times \frac{1}{4} = 18$$

6.The 7th and 10th terms of an arithmetic progression are 158 and 149 respectively.

(a) Find the common difference and the first term of the progression.

[3]

$a+6d=158$

$a+9d=149$

$-3d=9$

$d=-3$

$a+6d=158$

$a-18=158$

$a=176$

(b) Find the least number of terms of the progression for their sum to be negative.

[3]

$S_n<0$

$\frac{n}{2}(2a+(n-1)d)<0$

$\frac{n}{2}(352+(n-1)-3)<0$

$n(352-3n+3)<0$

$n(355-3n)<0$

$-3n<-355$

$n>118.\dot{3}$

$n=119$

7.(a) The sum of the first 4 terms of an arithmetic progression is 38 and the sum of the next 4 terms is 86. Find the first term and the common difference.

[5]

5+6+7+8

$$S_4 = 2(2a+3d)$$

$$38 = 2(2a+3d)$$

$$19 = 2a+3d \quad \text{— ①}$$

$$a+4d+a+5d+a+6d+a+7d = 86$$

$$4a+22d = 86$$

$$2a+11d = 43 \quad \text{— ②}$$

$$-\;2a - 3d = -19$$

$$8d = 24$$

$$d = 3$$

$$2a+9 = 19$$

$$2a = 10$$

$$a = 5$$

(b) The third term of a geometric progression is 12 and the sixth term is -96. Find the sum of the first 10 terms of this progression.

[6]

$ar^2 = 12 \qquad ar^5 = -96$

$a = \frac{12}{r^2} \qquad \frac{12}{r^2} \times r^5 = -96$

$r^3 = -8$

$r = -2$

$a = \frac{12}{4} = 3$

$$S_{10} = \frac{a(1-r^{10})}{1-r}$$

$$= \frac{3(1-(-2)^{10})}{1+2}$$

$$= \frac{\cancel{3}(1-1024)}{\cancel{3}}$$

$$= -1023$$